The Beurling operator for the hyperbolic plane

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Abstract. We find a Beurling operator for the hyperbolic plane, and obtain an L^2 norm identity for it, as well as L^p estimates.

1. Introduction and statement of main results

The Beurling transform. The Beurling transform (or operator) $\mathbf{B}: L^2(\mathbb{C}) \to L^2(\mathbb{C})$ is formally the operator $\mathbf{B} = \boldsymbol{\partial}\bar{\boldsymbol{\partial}}^{-1}$. Here, we use the notation

$$\boldsymbol{\partial}_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \quad \bar{\boldsymbol{\partial}}_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right),$$

and put $\boldsymbol{\Delta}_z = \boldsymbol{\partial}_z\bar{\boldsymbol{\partial}}_z$; as above, we frequently suppress the subscript z. This way of defining $\mathbf{B}$ leaves some ambiguity, as there are many possible ways to define $\bar{\boldsymbol{\partial}}^{-1}$. The choice is to use the Cauchy transform $\mathbf{C}$ for $\bar{\boldsymbol{\partial}}^{-1}$,

$$\mathbf{C}[f](z) = \int_{\mathbb{C}} \frac{f(w)}{z-w}\,\mathrm{d}A(w), \qquad z \in \mathbb{C},$$

where

$$\mathrm{d}A(z) = \frac{\mathrm{d}x\mathrm{d}y}{\pi}, \qquad z = x + \mathrm{i}y,$$

is normalized area measure. Unfortunately, the integral defining $\mathbf{C}[f]$ is not well-defined for all $f \in L^2(\mathbb{C})$, but if f is compactly supported, there is no problem. Differentiating the Cauchy transform, we get

$$\mathbf{B}[f](z) = -\mathrm{pv}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\,\mathrm{d}A(w), \qquad z \in \mathbb{C},$$

where "pv" stands for *principal value*. It is easy to show, using Fourier analysis or Green's formula, that $\mathbf{B}$ acts isometrically on $L^2(\mathbb{C})$:

$$\|\mathbf{B}[f]\|^2_{L^2(\mathbb{C})} = \|f\|^2_{L^2(\mathbb{C})} = \int_{\mathbb{C}} |f|^2\mathrm{d}A, \tag{1.1}$$

where the rightmost identity defines the norm in $L^2(\mathbb{C})$. It is well-known that $\mathbf{B}$ acts boundedly on $L^p(\mathbb{C})$ for $1 < p < +\infty$; let $B(p)$ denote its norm, that is, the best constant such that

$$\|\mathbf{B}[f]\|_{L^p(\mathbb{C})} \le B(p)\|f\|_{L^p(\mathbb{C})}, \qquad f \in L^p(\mathbb{C}),$$

holds. It is easy to show that there is an estimate from below as well:

$$\frac{1}{B(p)}\|f\|_{L^p(\mathbb{C})} \le \|\mathbf{B}[f]\|_{L^p(\mathbb{C})} \le B(p)\|f\|_{L^p(\mathbb{C})}, \qquad f \in L^p(\mathbb{C}). \tag{1.2}$$

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A well-known conjecture due to Tadeusz Iwaniec (see [7], [2], [6], [3]) claims that

$$B(p)=\max\left\{p-1,\frac{1}{p-1}\right\},\qquad 1<p<+\infty.$$

There is a formulation of (1.2) which does not use singular integrals:

$$\frac{1}{B(p)}\|\bar{\boldsymbol\partial}g\|_{L^p(\mathbb{C})}\le\|\boldsymbol\partial g\|_{L^p(\mathbb{C})}\le B(p)\|\bar{\boldsymbol\partial}g\|_{L^p(\mathbb{C})},\qquad g\in C_0^\infty(\mathbb{C}),\tag{1.3}$$

where $C_0^\infty(\mathbb{C})$ is the space of compactly supported test functions.

The Cauchy transform on a space of odd functions. Let $L^p(\mathbb{C},|\operatorname{Im} z|^p)$ denote the L^p space on $\mathbb{C}$ with norm

$$\|f\|^p_{L^p(\mathbb{C},|\operatorname{Im} z|^p)}=\int_{\mathbb{C}}|f(z)|^p|\operatorname{Im} z|^p\mathrm{d}A(z)<+\infty.$$

Moreover, let $L^p_{\mathrm{odd}}(\mathbb{C},|\operatorname{Im} z|^p)$ stand for the subspace of functions f subject to the symmetry condition

$$f(z)\equiv-f(\bar z).$$

Let $\mathbb{C}_+$ and $\mathbb{C}_-$ denote the upper and lower half-planes, respectively:

$$\mathbb{C}_+=\{z\in\mathbb{C}:\ \operatorname{Im} z>0\},\qquad \mathbb{C}_-=\{z\in\mathbb{C}:\ \operatorname{Im} z<0\}.$$

By invoking the Hardy inequality for the upper half-plane, one can show (for $1<p<+\infty$) that if $f\in L^p_{\mathrm{odd}}(\mathbb{C},|\operatorname{Im} z|^p)$, then $\mathbf{C}[f]\in L^p(\mathbb{C})$. We remark that the definition of $\mathbf{C}[f]$ for $f\in L^p_{\mathrm{odd}}(\mathbb{C},|\operatorname{Im} z|^p)$ is to be understood in a principal value sense, with respect to reflection in the real line. The precise optimal constant $C(p)$ such that

$$\|\mathbf{C}[f]\|_{L^p(\mathbb{C}_+)}\le C(p)\,\|f\|_{L^p(\mathbb{C},|\operatorname{Im} z|^p)},\qquad f\in L^p_{\mathrm{odd}}(\mathbb{C},|\operatorname{Im} z|^p),$$

appears to be unknown. For $p=2$, however, it is easy to see that $C(2)=4$.

The Beurling transform on the space of odd functions. We shall prove that for $1<p<+\infty$, the Beurling transform defines a bounded operator (understood in a principal value sense around the singularity as usual, but also with respect to reflection in the real line)

$$\mathbf{B}:\ L^p_{\mathrm{odd}}(\mathbb{C},|\operatorname{Im} z|^p)\to L^p(\mathbb{C},|\operatorname{Im} z|^p),$$

and that for $p=2$, there is an associated norm identity. More precisely, we shall obtain the following.

Theorem 1.1. $(1<p<+\infty)$ *Let $B(p)$ equal the norm of $\mathbf{B}:\ L^p(\mathbb{C})\to L^p(\mathbb{C})$. Then we have, for $f\in L^p_{\mathrm{odd}}(\mathbb{C},|\operatorname{Im} z|^p)$, the two-sided estimate*

$$\begin{aligned}B(p)^{-p}&\int_{\mathbb{C}_+}\big|(\operatorname{Im} z)f(z)+\tfrac{\mathrm{i}}{2}\big(\mathbf{C}[f](z)-\mathbf{C}[f](\bar z)\big)\big|^p\mathrm{d}A(z)\\&\le\int_{\mathbb{C}_+}|\mathbf{B}[f](z)|^p(\operatorname{Im} z)^p\mathrm{d}A(z)\le B(p)^p\int_{\mathbb{C}_+}\big|(\operatorname{Im} z)f(z)+\tfrac{\mathrm{i}}{2}\big(\mathbf{C}[f](z)-\mathbf{C}[f](\bar z)\big)\big|^2\mathrm{d}A(z).\end{aligned}$$

As $B(2)=1$, this becomes a norm identity for $p=2$.

This theorem has an interpretation in terms of a hyperbolic plane Beurling transform, as will be explained in the next section.

Notation. We shall at times need conjugate symbol operators, as defined by

$$\bar{\mathbf{T}}[f]=\operatorname{conj}(\mathbf{T}[\bar f]),$$

and we apply this notational convention to all the operators considered here.

2. Beurling transform for the hyperbolic plane

The hyperbolic plane and differential operators. Let $\mathbb{H}$ denote the hyperbolic plane; we shall use the model

$$\mathbb{H} = \langle \mathbb{C}_+, \mathrm{d}s_{\mathbb{H}} \rangle,$$

where $\mathbb{C}_+$ is the upper half plane, and

$$\mathrm{d}s_{\mathbb{H}}(z) = \frac{|\mathrm{d}z|}{\mathrm{Im}\, z}$$

is the Poincaré metric. The hyperbolic area element is

$$\mathrm{d}A_{\mathbb{H}}(z) = \frac{\mathrm{d}A(z)}{(\mathrm{Im}\, z)^2},$$

and we write $L^2(\mathbb{H})$ for the L^2 space with norm

$$\|f\|^2_{L^2(\mathbb{H})} = \int_{\mathbb{H}} |f(z)|^2 \mathrm{d}A_{\mathbb{H}}(z) = \int_{\mathbb{C}_+} |f(z)|^2 \frac{\mathrm{d}A(z)}{(\mathrm{Im}\, z)^2}.$$

This is the L^2 space of the hyperbolic plane. More, generally, for $1 < p < +\infty$, we write $L^p(\mathbb{H})$ for the L^p space with norm

$$\|f\|^p_{L^p(\mathbb{H})} = \int_{\mathbb{C}_+} |f(z)|^p \frac{\mathrm{d}A(z)}{(\mathrm{Im}\, z)^p}.$$

This space is not in general the L^p space with respect to the hyperbolic area element, but hopefully this will not cause any misunderstanding. Let $\mathbf{M}$ denote the multiplication operator

$$\mathbf{M}[f](z) = (\mathrm{Im}\, z) f(z).$$

Then $\mathbf{M} : L^p(\mathbb{C}_+) \to L^p(\mathbb{H})$ is a surjective isometry. Associated with the half-plane model of $\mathbb{H}$, we have the geometrically induced differential operators $\boldsymbol{\partial}^{\uparrow}, \bar{\boldsymbol{\partial}}^{\uparrow}$:

$$\boldsymbol{\partial}^{\uparrow} = \mathbf{M}\boldsymbol{\partial}, \quad \bar{\boldsymbol{\partial}}^{\uparrow} = \mathbf{M}\bar{\boldsymbol{\partial}}.$$

After all, the length scale on $\mathbb{C}_+$ should be modified to correspond to that of $\mathbb{H}$. There are also the "dual" geometrically induced differential operators $\boldsymbol{\partial}^{\downarrow}, \bar{\boldsymbol{\partial}}^{\downarrow}$:

$$\boldsymbol{\partial}^{\downarrow} = \mathbf{M}^2\boldsymbol{\partial}\mathbf{M}^{-1}, \quad \bar{\boldsymbol{\partial}}^{\downarrow} = \mathbf{M}^2\bar{\boldsymbol{\partial}}\mathbf{M}^{-1},$$

with the properties that

$$\langle \boldsymbol{\partial}^{\uparrow} f, g \rangle_{L^2(\mathbb{H})} = -\langle f, \bar{\boldsymbol{\partial}}^{\downarrow} g \rangle_{L^2(\mathbb{H})}, \quad \langle \bar{\boldsymbol{\partial}}^{\uparrow} f, g \rangle_{L^2(\mathbb{H})} = -\langle f, \boldsymbol{\partial}^{\downarrow} g \rangle_{L^2(\mathbb{H})},$$

provided at least one of f, g is in the class $C_0^\infty(\mathbb{C}_+)$ of compactly supported test functions, and the other is, say, in $L^2(\mathbb{H})$ (the partial derivatives are interpreted in the sense of distribution theory when necessary). The hyperbolic Laplacian $\boldsymbol{\Delta}_{\mathbb{H}}$ is obtained as a combination of two such geometric differential operators:

$$\boldsymbol{\Delta}_{\mathbb{H}} = \bar{\boldsymbol{\partial}}^{\downarrow}\boldsymbol{\partial}^{\uparrow} = \boldsymbol{\partial}^{\downarrow}\bar{\boldsymbol{\partial}}^{\uparrow} = \mathbf{M}^2\boldsymbol{\Delta}.$$

Hyperbolic plane Beurling operators. In analogy with the planar Beurling transform $\mathbf{B} = \boldsymbol{\partial}\bar{\boldsymbol{\partial}}^{-1}$, we propose for the hyperbolic plane $\boldsymbol{\partial}^{\downarrow}(\bar{\boldsymbol{\partial}}^{\downarrow})^{-1}$ as a candidate for the title "hyperbolic plane Beurling transform". Again, there is the matter of the choice of $(\bar{\boldsymbol{\partial}}^{\downarrow})^{-1}$. In contrast with the $\bar{\boldsymbol{\partial}}$-problem in the plane, given a function $f \in L^2(\mathbb{H})$, there always exists a solution $u \in L^2(\mathbb{H})$ with $\bar{\boldsymbol{\partial}}^{\downarrow} u = f$, and

$$\|u\|_{L^2(\mathbb{H})} \le 4\|f\|_{L^2(\mathbb{H})}.$$

This follows from the well-known Hardy inequality in a manner which will be explained in a later section. In particular, there always exists a unique solution $u = u_f$ of minimal norm in $L^2(\mathbb{H})$. We write $u_f = [\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min} f$ for this minimal solution, and have thus defined the operator $[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}$. In a similar manner, we may define the operator $[\boldsymbol{\partial}^{\downarrow}]^{-1}_{\min}$. It turns out that both operators $[\boldsymbol{\partial}^{\downarrow}]^{-1}_{\min}$

and $[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}$ act boundedly on $L^p(\mathbb{H})$ for $1 < p < +\infty$. Our candidate Beurling operator for the hyperbolic plane is $\boldsymbol{\partial}^{\downarrow}[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}$. Theorem 1.1 may be formulated in these terms.

Theorem 2.1. $(1 < p < +\infty)$ *For $p = 2$, we have the norm identity*

$$\big\|\boldsymbol{\partial}^{\downarrow}[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}f\big\|_{L^2(\mathbb{H})} = \big\|f + \tfrac{\mathrm{i}}{2}\big([\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}f + [\boldsymbol{\partial}^{\downarrow}]^{-1}_{\min}f\big)\big\|_{L^2(\mathbb{H})}, \qquad f \in L^2(\mathbb{H}),$$

while for general p, we have

$$\frac{1}{B(p)}\big\|f + \tfrac{\mathrm{i}}{2}\big([\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}f + [\boldsymbol{\partial}^{\downarrow}]^{-1}_{\min}f\big)\big\|_{L^p(\mathbb{H})} \le \big\|\boldsymbol{\partial}^{\downarrow}[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}f\big\|_{L^p(\mathbb{H})}$$
$$\le B(p)\big\|f + \tfrac{\mathrm{i}}{2}\big([\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}f + [\boldsymbol{\partial}^{\downarrow}]^{-1}_{\min}f\big)\big\|_{L^p(\mathbb{H})}, \qquad f \in L^p(\mathbb{H}).$$

Remark 2.2. (a) The additional term

$$\tfrac{\mathrm{i}}{2}\big([\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}f + [\boldsymbol{\partial}^{\downarrow}]^{-1}_{\min}f\big)$$

in the norm on the right hand side of the norm identity should be interpreted as a correction due to curvature.
(b) The hyperbolic plane $\mathbb{H}$ is the universal covering surface of a number of Riemann surfaces. The study of the operator $\boldsymbol{\partial}^{\downarrow}[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}$ on the hyperbolic plane should lead to an understanding of the same issue on those Riemann surfaces.

3. Consequences of the Hardy inequality

Function spaces. We recall that $f \in L^p(\mathbb{H})$ means that f is in the Lebesgue space over $\mathbb{C}_+$ with

$$\|f\|_{L^p(\mathbb{H})} = \|\mathbf{M}^{-1}[f]\|_{L^p(\mathbb{C}_+)} < +\infty.$$

Here, as usual, $\mathbf{M}[f](z) = (\operatorname{Im} z)f(z)$. We shall also need the space $L^p(\mathbb{H}^*)$, the Lebesgue space over $\mathbb{C}_+$ consisting of functions f with

$$\|f\|_{L^p(\mathbb{H}^*)} = \|\mathbf{M}[f]\|_{L^p(\mathbb{C}_+)} < +\infty.$$

The way things are set up, $\mathbf{M}$ acts isometrically and surjectively in the following instances:

$$\mathbf{M} : L^p(\mathbb{H}^*) \to L^p(\mathbb{C}_+), \quad \mathbf{M} : L^p(\mathbb{C}_+) \to L^p(\mathbb{H}), \quad \mathbf{M}^2 : L^p(\mathbb{H}^*) \to L^p(\mathbb{H}).$$

Hardy's inequality for the upper half plane. Let $\mathbb{C}_+$ denote the open upper half plane. By Hardy's inequality for the upper half space,

$$\int_{\mathbb{C}_+} |f(z)|^p \frac{\mathrm{d}A(z)}{(\operatorname{Im} z)^p} \le 2^{p/2}(1-1/p)^{-p}\int_{\mathbb{C}_+}\big(|\boldsymbol{\partial}f(z)|^2 + |\bar{\boldsymbol{\partial}}f(z)|^2\big)^{p/2}\mathrm{d}A(z), \tag{3.1}$$

for $f \in C_0^\infty(\mathbb{C}_+)$. The constant is sharp (see, e. g., [5], [9]). If we use that for $a, b \in \mathbb{C}$,

$$(|a|^2 + |b|^2)^{p/2} \le A(p)(|a|^p + |b|^p), \qquad A(p) = \max\{1, 2^{-1+p/2}\},$$

we get

$$\int_{\mathbb{C}_+} |f(z)|^p \frac{\mathrm{d}A(z)}{(\operatorname{Im} z)^p} \le 2^{p/2}(1-1/p)^{-p}A(p)\int_{\mathbb{C}_+}\big(|\boldsymbol{\partial}f(z)|^p + |\bar{\boldsymbol{\partial}}f(z)|^p\big)\mathrm{d}A(z), \tag{3.2}$$

and since by (1.2),

$$\int_{\mathbb{C}_+} |\boldsymbol{\partial}f(z)|^p\mathrm{d}A(z) \le B(p)^p\int_{\mathbb{C}_+}|\bar{\boldsymbol{\partial}}f(z)|^p\mathrm{d}A(z),$$

we obtain from (3.2) that

$$\int_{\mathbb{C}_+} |f(z)|^p \frac{\mathrm{d}A(z)}{(\operatorname{Im} z)^p} \le 2^{p/2}(1-1/p)^{p}A(p)(1+B(p)^p)\int_{\mathbb{C}_+}|\bar{\boldsymbol{\partial}}f(z)|^p\mathrm{d}A(z) \tag{3.3}$$

for $f \in C_0^\infty(\mathbb{C}_+)$. In particular, for $p=2$ we get

$$\int_{\mathbb{C}_+} |f(z)|^2 \frac{\mathrm{d}A(z)}{(\operatorname{Im} z)^2} \le 16 \int_{\mathbb{C}_+} |\bar{\boldsymbol{\partial}} f(z)|^2 \mathrm{d}A(z), \qquad f \in C_0^\infty(\mathbb{C}_+). \tag{3.4}$$

Here, the constant 16 is sharp.

Cauchy-type operators. We put

$$\mathbf{C}^\downarrow[g](z) = \int_{\mathbb{C}_+} \left(\frac{1}{z-w} - \frac{1}{z-\bar{w}} \right) g(w)\, \mathrm{d}A(w) = 2\mathrm{i} \int_{\mathbb{C}_+} \frac{g(w)\, \operatorname{Im} w}{(z-w)(z-\bar{w})}\, \mathrm{d}A(w),$$

and

$$\mathbf{C}^\uparrow[g](z) = \int_{\mathbb{C}_+} \left(\frac{1}{z-w} - \frac{1}{\bar{z}-w} \right) g(w)\, \mathrm{d}A(w) = -2\mathrm{i} \operatorname{Im} z \int_{\mathbb{C}_+} \frac{g(w)}{(z-w)(\bar{z}-w)}\, \mathrm{d}A(w),$$

for all locally integrable functions g for which the integrals make sense (almost everywhere on $\mathbb{C}_+$). To understand the action of $\mathbf{C}^\uparrow$, we note that

$$\begin{aligned} f(z) - \mathbf{C}^\uparrow[\bar{\boldsymbol{\partial}} f](z) &= -\int_{\mathbb{C}_+} \left(\frac{1}{z-w} - \frac{1}{\bar{z}-w} \right) \bar{\partial} f(w)\, \mathrm{d}A(w) \\ = \int_{\mathbb{C}_+} \bar{\partial}_w \left\{ \left(\frac{1}{w-z} - \frac{1}{w-\bar{z}} \right) f(w) \right\} \mathrm{d}A(w) &= \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \left(\frac{1}{w-z} - \frac{1}{w-\bar{z}} \right) f(w)\, \mathrm{d}w, \qquad z \in \mathbb{C}_+, \end{aligned} \tag{3.5}$$

provided f and $\bar{\boldsymbol{\partial}} f$ are smooth and drop off relatively quickly to 0 at infinity (the middle integral is to be interpreted in the sense of distributions theory). As a first application of (3.5), we find that

$$\mathbf{C}^\uparrow[\bar{\boldsymbol{\partial}} f] = f, \qquad f \in C_0^\infty(\mathbb{C}_+). \tag{3.6}$$

Let $A^2(\mathbb{C}_+)$ and $\operatorname{conj}(A^2(\mathbb{C}_+))$ denote the closed subspaces of $L^2(\mathbb{C}_+)$ consisting of the holomorphic and conjugate-holomorphic functions, respectively. In $L^2(\mathbb{C}_+)$, the closure of $\bar{\boldsymbol{\partial}} C_0^\infty(\mathbb{C}_+)$ equals $L^2(\mathbb{C}_+) \ominus \operatorname{conj}(A^2(\mathbb{C}_+))$ (this fact is known as Havin's lemma). A second application of (3.5) shows that

$$\mathbf{C}^\uparrow[g] = 0, \qquad g \in \operatorname{conj}(A^2(\mathbb{C}_+)), \tag{3.7}$$

which means that we have determined the action of $\mathbf{C}^\uparrow$ on all of $L^2(\mathbb{C}_+)$. In particular,

$$\int_{\mathbb{C}_+} |\mathbf{C}^\uparrow[g](z)|^2 \frac{\mathrm{d}A(z)}{(\operatorname{Im} z)^2} \le 16 \int_{\mathbb{C}_+} |g(z)|^2 \mathrm{d}A(z), \qquad g \in L^2(\mathbb{C}_+). \tag{3.8}$$

Expressed differently, the operator

$$\mathbf{C}^\uparrow : L^2(\mathbb{C}_+) \to L^2(\mathbb{H}) \tag{3.9}$$

is bounded and has norm 4. A similar argument based on (3.3) shows that

$$\mathbf{C}^\uparrow : L^p(\mathbb{C}_+) \to L^p(\mathbb{H}), \qquad 1 < p < +\infty,$$

with a norm bound which depends on p. Let $C^\uparrow(p)$ be the norm of this operator, that is, the smallest bound such that

$$\|\mathbf{C}^\uparrow[g]\|_{L^2(\mathbb{H})} \le C^\uparrow(p) \|g\|_{L^p(\mathbb{C}_+)}, \qquad g \in L^p(\mathbb{C}_+).$$

Likewise, we let $C^\downarrow(p)$ denote the norm of the operator

$$\mathbf{C}^\downarrow : L^p(\mathbb{H}^*) \to L^p(\mathbb{C}_+), \qquad 1 < p < +\infty.$$

We introduce the operators $\mathbf{D}^\uparrow$, $\mathbf{D}^\downarrow$, as given by

$$\mathbf{D}^\uparrow[g](z) = \int_{\mathbb{C}_+} \frac{g(w)}{(z-w)(\bar{z}-w)}\, \mathrm{d}A(w), \qquad z \in \mathbb{C}_+,$$

and

$$\mathbf{D}^\downarrow[g](z) = \int_{\mathbb{C}_+} \frac{g(w)}{(z-w)(z-\bar{w})}\, \mathrm{d}A(w), \qquad z \in \mathbb{C}_+.$$

We readily check that

$$\mathbf{C}^{\uparrow} = -2\mathrm{i}\,\mathbf{M}\mathbf{D}^{\uparrow}, \qquad \mathbf{C}^{\downarrow} = 2\mathrm{i}\,\mathbf{D}^{\downarrow}\mathbf{M}, \tag{3.10}$$

where we recall that $\mathbf{M}[f](z) = (\operatorname{Im} z)f(z)$. The analogous operators in the setting of the unit disk $\mathbb{D}$ in place of $\mathbb{C}_+$ appeared recently in [4]. Let $D^{\uparrow}(p)$ denote the norm of the operator $\mathbf{D}^{\uparrow} : L^p(\mathbb{C}_+) \to L^p(\mathbb{C}_+)$, and let $D^{\downarrow}(p)$ have the analogous meaning of the norm of $\mathbf{D}^{\downarrow} : L^p(\mathbb{C}_+) \to L^p(\mathbb{C}_+)$. The norms $C^{\uparrow}(p)$, $C^{\downarrow}(p)$, $D^{\uparrow}(p)$, and $D^{\downarrow}(p)$ are all connected:

$$C^{\uparrow}(p) = 2D^{\uparrow}(p), \quad C^{\downarrow}(p) = 2D^{\downarrow}(p), \quad D^{\downarrow}(p) = D^{\uparrow}(p'),$$

where $p' = p/(p-1)$ is the dual exponent $(1 < p < +\infty)$. By duality, the information on the null space of $\mathbf{C}^{\uparrow}$ supplied by (3.7) leads to information on the range of $\mathbf{C}^{\downarrow}$:

$$\mathbf{C}^{\downarrow} : L^2(\mathbb{H}^*) \to L^2(\mathbb{C}_+) \ominus A^2(\mathbb{C}_+).$$

The operator $\mathbf{C}^{\downarrow}$ therefore furnishes the least norm solution to the $\bar{\boldsymbol{\partial}}$ problem: $u = \mathbf{C}^{\downarrow}[f]$ has smallest norm in $L^2(\mathbb{C}_+)$ among all solutions to

$$\bar{\boldsymbol{\partial}} u = f(z), \qquad z \in \mathbb{C}_+.$$

This means that we may express the operator $[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}$ encountered in the introduction in terms of $\mathbf{C}^{\downarrow}$ or $\mathbf{D}^{\downarrow}$:

$$[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min} = \mathbf{M}\mathbf{C}^{\downarrow}\mathbf{M}^{-2} = 2\mathrm{i}\,\mathbf{M}\mathbf{D}^{\downarrow}\mathbf{M}^{-1} :\ L^2(\mathbb{H}) \to L^2(\mathbb{H}). \tag{3.11}$$

Beurling-type operators. We introduce the Beurling-type operators

$$\mathbf{B}^{\downarrow}[f](z) = \boldsymbol{\partial}\mathbf{C}^{\downarrow}[f](z) = \mathrm{pv}\int_{\mathbb{C}_+}\left[\frac{1}{(z-\bar{w})^2} - \frac{1}{(z-w)^2}\right] f(w)\,\mathrm{d}A(w), \qquad z \in \mathbb{C}_+,$$

and

$$\mathbf{B}^{\uparrow}[f](z) = \mathrm{pv}\int_{\mathbb{C}_+}\left[\frac{1}{(\bar{z}-w)^2} - \frac{1}{(z-w)^2}\right] f(w)\,\mathrm{d}A(w), \qquad z \in \mathbb{C}_+,$$

for functions f such that the above expressions make sense. It follows from (3.11) that the operator $\boldsymbol{\partial}^{\downarrow}[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min}$ from the introduction may be written

$$\boldsymbol{\partial}^{\downarrow}[\bar{\boldsymbol{\partial}}^{\downarrow}]^{-1}_{\min} = \mathbf{M}^2\mathbf{B}^{\downarrow}\mathbf{M}^{-2}.$$

4. Liouville's theorem for the hyperbolic plane

Liouville's theorem revisited. Liouville's theorem states that the only bounded harmonic functions in the complex plane are the constants. If we ask the functions to be in $L^p(\mathbb{C})$ as well, the only harmonic function is the constant 0. In the hyperbolic plane things are quite different, as there are plenty of bounded harmonic functions in $\mathbb{H}$. However, there are not many harmonic functions in $L^p(\mathbb{H})$.

Theorem 4.1. (Liouville for $\mathbb{H}$) *Suppose $1 \le p < +\infty$. If a function $f \in L^p(\mathbb{H})$ is harmonic in $\mathbb{H}$, then $f = 0$.*

Proof. Our first observation, which relies on the assumption $1 \le p < +\infty$, is that

f is harmonic $\implies$ $|f|^p$ is subharmonic.

Since f is assumed harmonic in $\mathbb{C}_+$, then $|f(z+t)|^p$ is subharmonic in $\mathbb{C}_+$ for each fixed $t \in \mathbb{R}$. In particular, the integrals

$$\int_{-\infty}^{+\infty} |f(z+t)|^p \mathrm{d}t$$

are subharmonic in $\mathbb{C}_+$ (in the extended sense, which allows for $+\infty$ as a subharmonic function), and only depend on the imaginary part of z. If we put

$$M_p(y)=\int_{-\infty}^{+\infty}|f(t+\mathrm{i}y)|^p\mathrm{d}t,\qquad 0<y<+\infty,$$

we conclude that M_p is a convex function. That $f\in L^p(\mathbb{H})$ means that

$$\|f\|^p_{L^p(\mathbb{H})}=\int_{\mathbb{C}_+}\frac{|f(z)|^p}{|\operatorname{Im} z|^p}\mathrm{d}A(z)=\frac{1}{\pi}\int_0^{+\infty}\frac{M_p(y)}{y^p}\mathrm{d}y<+\infty. \tag{4.1}$$

The function M_p being convex, the limit

$$M_p(0^+)=\lim_{y\to 0}M_p(0)$$

exists. If $M_p(0^+)>0$, the above integral (4.1) cannot converge. It remains to consider $M_p(0^+)=0$. In this case, unless $M_p(y)$ vanishes everywhere, it must grow at least linearly in y for big y. But then the integral (4.1) diverges for $1\le p\le 2$. So, for $1\le p\le 2$, we are done.

We turn to the remaining case $2<p<+\infty$. Again, if $M_p(0^+)>0$, the integral (4.1) cannot converge. It remains to consider $M_p(0^+)=0$. From the definition of M_p, we realize that this means that the harmonic function f has boundary value 0 on the real line $\mathbb{R}$ in every reasonable sense. By Schwarz reflection, f extends harmonically to all of $\mathbb{C}$, the extension to the lower half plane $\mathbb{C}_-$ being supplied by $-f(\bar z)$. We denote this extension by f as well. We clearly have

$$\int_{\mathbb{C}}\frac{|f(z)|^p}{|z|^p}\mathrm{d}A(z)\le\int_{\mathbb{C}}\frac{|f(z)|^p}{|\operatorname{Im} z|^p}\mathrm{d}A(z)=2\|f\|^p_{L^p(\mathbb{H})}<+\infty,$$

and if $\mathbb{D}(0,R)$ denotes the open disk about the origin with radius R, we see that

$$\int_{\mathbb{D}(0,R)}\frac{|f(z)|^2}{|z|^2}\mathrm{d}A(z)\le\left\{2\|f\|^p_{L^p(\mathbb{H})}\right\}^{2/p}R^{2-4/p}. \tag{4.2}$$

It is possible to calculate the reproducing kernel for the space of harmonic functions that vanish at the origin with norm defined by the left hand side of (4.2). The expression for the reproducing kernel allows us to estimate the gradient:

$$|\nabla f(z)|^2\le CR^{-2}(1-|z|^2/R^2)^{-4}\int_{\mathbb{D}(0,R)}\frac{|f(w)|^2}{|w|^2}\mathrm{d}A(w),\qquad z\in\mathbb{D}(0,R),$$

where C is an appropriate constant. When we combine this estimate with (4.2), we get

$$|\nabla f(z)|^2\le 2^{2/p}C\|f\|^2_{L^p(\mathbb{H})}R^{-4/p}(1-|z|^2/R^2)^{-4},\qquad z\in\mathbb{D}(0,R).$$

As $R\to+\infty$, the right hand side tends to 0, and so $\nabla f=0$ throughout $\mathbb{C}$. It follows that f is constant, and that constant must equal 0. The proof is complete. □

5. Main results

The norm estimate of the hyperbolic plane Beurling transform. We reformulate Theorem 1.1 in terms of the operators $\mathbf{C}^{\downarrow}$, $\bar{\mathbf{C}}^{\downarrow}$, and $\mathbf{B}^{\downarrow}$.

Theorem 5.1. $(1<p<+\infty)$ *The operators*

$$\mathbf{B}^{\downarrow}:\,L^p(\mathbb{H}^*)\to L^p(\mathbb{H}^*)$$

and

$$\mathbf{B}^{\uparrow}:\,L^p(\mathbb{H})\to L^p(\mathbb{H})$$

are bounded. Indeed, we have the norm estimate

$$B(p)^{-1}\left\|\mathbf{M}[f]+\tfrac{\mathrm{i}}{2}\left[\mathbf{C}^{\downarrow}[f]+\bar{\mathbf{C}}^{\downarrow}[f]\right]\right\|^2_{L^p(\mathbb{C}_+)}\le\|\mathbf{M}\mathbf{B}^{\downarrow}[f]\|^2_{L^p(\mathbb{C}_+)}\le B(p)\left\|\mathbf{M}[f]+\tfrac{\mathrm{i}}{2}\left[\mathbf{C}^{\downarrow}[f]+\bar{\mathbf{C}}^{\downarrow}[f]\right]\right\|^2_{L^p(\mathbb{C}_+)}$$

for all $f \in L^p(\mathbb{H}^*)$. Here, $B(p)$ denotes the norm of the Beurling transform on $L^p(\mathbb{C})$.

Before we turn to the proof of the theorem, we note that we have the commutator relationships (n is an integer)

$$\boldsymbol{\partial}\mathbf{M}^n - \mathbf{M}^n\boldsymbol{\partial} = -\frac{\mathrm{i}n}{2}\,\mathbf{M}^{n-1}, \quad \bar{\boldsymbol{\partial}}\mathbf{M}^n - \mathbf{M}^n\bar{\boldsymbol{\partial}} = \frac{\mathrm{i}n}{2}\,\mathbf{M}^{n-1}. \tag{5.1}$$

Proof of Theorem 5.1. As we saw in the previous section, it is a consequence of the Hardy inequality (3.3) that $\mathbf{C}^{\downarrow}[f]$ and $\bar{\mathbf{C}}^{\downarrow}[f]$ are in $L^p(\mathbb{C}_+)$ provided that $f \in L^p(\mathbb{H}^*)$.

Next, we shall assume $f \in L^p(\mathbb{H}^*)$ is of the form $f = \boldsymbol{\Delta} F$, for some $F \in C_0^\infty(\mathbb{C}_+)$. We first claim that the collection of such f is dense in $L^p(\mathbb{H}^*)$. To this end, suppose $g \in L^{p'}(\mathbb{C}_+)$ is such that

$$\int_{\mathbb{C}_+} (\operatorname{Im} z)\boldsymbol{\Delta} F(z)\,\bar{g}(z)\,\mathrm{d}A(z) = 0.$$

By Green's formula, we get, in the sense of distribution theory,

$$\int_{\mathbb{C}_+} F(z)\,\boldsymbol{\Delta}((\operatorname{Im} z)\bar{g}(z))\,\mathrm{d}A(z) = 0,$$

for all $F \in C_0^\infty(\mathbb{C}_+)$. It follows that

$$\boldsymbol{\Delta}\mathbf{M}[g] = 0,$$

so that $\mathbf{M}[g] \in L^{p'}(\mathbb{H})$ is harmonic. By Theorem 4.1, $g = 0$, and the claim follows.

In terms of the function F, we have

$$\mathbf{C}^{\downarrow}[f] = \boldsymbol{\partial} F, \qquad \bar{\mathbf{C}}^{\downarrow}[f] = \bar{\boldsymbol{\partial}} F.$$

We now see that the norm estimate of the theorem follows once it has been established that

$$\begin{aligned} B(p)^{-1}\big\|\mathbf{M}\boldsymbol{\Delta} F + \tfrac{\mathrm{i}}{2}\,(\boldsymbol{\partial} F + \bar{\boldsymbol{\partial}} F)\big\|^2_{L^p(\mathbb{C}_+)} \\ \le \|\mathbf{M}\boldsymbol{\partial}^2 F\|^2_{L^p(\mathbb{C}_+)} \le B(p)\big\|\mathbf{M}\boldsymbol{\Delta} F + \tfrac{\mathrm{i}}{2}\,(\boldsymbol{\partial} F + \bar{\boldsymbol{\partial}} F)\big\|^2_{L^p(\mathbb{C}_+)}. \end{aligned} \tag{5.2}$$

To this end, we introduce the auxiliary function

$$G = \mathbf{M}^2\boldsymbol{\partial}\mathbf{M}^{-1}[F] \in C_0^\infty(\mathbb{C}_+).$$

We may think of $C_0^\infty(\mathbb{C}_+)$ as a subspace of $C_0^\infty(\mathbb{C})$ by extending the functions to vanish where they were previously undefined. In particular, it follows from (1.3) that

$$\frac{1}{B(p)}\|\bar{\boldsymbol{\partial}} G\|_{L^p(\mathbb{C}_+)} \le \|\boldsymbol{\partial} G\|_{L^p(\mathbb{C}_+)} \le B(p)\|\bar{\boldsymbol{\partial}} G\|_{L^p(\mathbb{C})_+}. \tag{5.3}$$

We first calculate $\boldsymbol{\partial} G$, using (5.1):

$$\begin{aligned} \boldsymbol{\partial} G = \boldsymbol{\partial}\mathbf{M}^2\boldsymbol{\partial}\mathbf{M}^{-1}[F] &= (\mathbf{M}^2\boldsymbol{\partial} - \mathrm{i}\mathbf{M})\boldsymbol{\partial}\mathbf{M}^{-1}[F] = (\mathbf{M}^2\boldsymbol{\partial} - \mathrm{i}\mathbf{M})(\mathbf{M}^{-1}\boldsymbol{\partial} + \tfrac{\mathrm{i}}{2}\mathbf{M}^{-2})[F] \\ &= \mathbf{M}^2\boldsymbol{\partial}\mathbf{M}^{-1}\boldsymbol{\partial} F + \tfrac{\mathrm{i}}{2}\mathbf{M}^2\boldsymbol{\partial}\mathbf{M}^{-2}[F] - \mathrm{i}\boldsymbol{\partial} + \tfrac{1}{2}\mathbf{M}^{-1}[F] \\ &= \mathbf{M}^2(\tfrac{\mathrm{i}}{2}\mathbf{M}^{-2} + \mathbf{M}^{-1}\boldsymbol{\partial})\boldsymbol{\partial} F + \tfrac{\mathrm{i}}{2}\mathbf{M}^2(\mathrm{i}\mathbf{M}^{-3} + \mathbf{M}^{-2}\boldsymbol{\partial})F - \mathrm{i}\boldsymbol{\partial} F + \tfrac{1}{2}\mathbf{M}^{-1}[F] \\ &= \tfrac{\mathrm{i}}{2}\boldsymbol{\partial} F + \mathbf{M}\boldsymbol{\partial}^2 F - \tfrac{1}{2}\mathbf{M}^{-1}[F] + \tfrac{\mathrm{i}}{2}\boldsymbol{\partial} F - \mathrm{i}\boldsymbol{\partial} F + \tfrac{1}{2}\mathbf{M}^{-1}[F] = \mathbf{M}\boldsymbol{\partial}^2 F. \end{aligned}$$

We next calculate $\bar{\boldsymbol{\partial}} G$:

$$\begin{aligned} \bar{\boldsymbol{\partial}} G = \bar{\boldsymbol{\partial}}\mathbf{M}^2\boldsymbol{\partial}\mathbf{M}^{-1}[F] &= (\mathbf{M}^2\bar{\boldsymbol{\partial}} + \mathrm{i}\mathbf{M})\boldsymbol{\partial}\mathbf{M}^{-1}[F] = (\mathbf{M}^2\bar{\boldsymbol{\partial}} + \mathrm{i}\mathbf{M})(\mathbf{M}^{-1}\boldsymbol{\partial} + \tfrac{\mathrm{i}}{2}\mathbf{M}^{-2})[F] \\ &= \mathbf{M}^2\bar{\boldsymbol{\partial}}\mathbf{M}^{-1}\boldsymbol{\partial} F + \tfrac{\mathrm{i}}{2}\mathbf{M}^2\bar{\boldsymbol{\partial}}\mathbf{M}^{-2}[F] + \mathrm{i}\boldsymbol{\partial} F - \tfrac{1}{2}\mathbf{M}^{-1}[F] \\ &= \mathbf{M}^2(-\tfrac{\mathrm{i}}{2}\mathbf{M}^{-2} + \mathbf{M}^{-1}\bar{\boldsymbol{\partial}})\boldsymbol{\partial} F + \tfrac{\mathrm{i}}{2}\mathbf{M}^2(-\mathrm{i}\mathbf{M}^{-3} + \mathbf{M}^{-2}\bar{\boldsymbol{\partial}})F + \mathrm{i}\boldsymbol{\partial} F - \tfrac{1}{2}\mathbf{M}^{-1}[F] \\ &= -\tfrac{\mathrm{i}}{2}\boldsymbol{\partial} F + \mathbf{M}\boldsymbol{\Delta} F + \tfrac{1}{2}\mathbf{M}^{-1}[F] + \tfrac{\mathrm{i}}{2}\bar{\boldsymbol{\partial}} F + \mathrm{i}\boldsymbol{\partial} F - \tfrac{1}{2}\mathbf{M}^{-1}[F] = \mathbf{M}\boldsymbol{\Delta} F + \tfrac{\mathrm{i}}{2}(\boldsymbol{\partial} F + \bar{\boldsymbol{\partial}} F). \end{aligned}$$

The claimed estimate (5.2) is now an immediate consequence of (5.3). The proof is complete. □

Remark 5.2. Let $\mathbf{E}$ denote the integral operator

$$\mathbf{E}[f](z) = \int_{\mathbb{C}_+} \frac{\mathrm{Re}(z-w)}{|(z-w)(z-\bar{w})|^2} f(w)\,\mathrm{d}A(w), \qquad z \in \mathbb{C}_+.$$

One shows that

$$\frac{1}{2}(\mathbf{C}^{\downarrow} + \bar{\mathbf{C}}^{\downarrow}) = 4\,\mathbf{MEM},$$

which makes $\mathbf{E} : L^2(\mathbb{C}_+) \to L^2(\mathbb{H}^*)$ a contraction. By duality, $\mathbf{E}$ acts as a contraction $L^2(\mathbb{H}) \to L^2(\mathbb{C}_+)$ as well. In terms of $\mathbf{E}$, the norm estimate of Theorem 5.1 may be written in the form

$$B(p)^{-1}\big\|f + 4\mathrm{i}\,\mathbf{ME}[f]\big\|_{L^p(\mathbb{C}_+)} \le \big\|\mathbf{MB}^{\downarrow}\mathbf{M}^{-1}[f]\big\|_{L^p(\mathbb{C}_+)} \le B(p)\big\|f + 4\mathrm{i}\,\mathbf{ME}[f]\big\|_{L^p(\mathbb{C}_+)},$$

where $f \in L^p(\mathbb{C}_+)$ is arbitrary.

6. Analysis of an operator

The operator. In the context of Theorem 5.1, with $p = 2$, we would like to study the operator

$$\mathbf{M} + \tfrac{\mathrm{i}}{2}\big[\mathbf{C}^{\downarrow} + \bar{\mathbf{C}}^{\downarrow}\big] : L^2(\mathbb{H}^*) \to L^2(\mathbb{C}_+)$$

with respect to range and null space. We first look at the range. It is a curious fact that this problem is intimately connected with the classical Whittaker (or Kummer) ordinary differential equation (see, e. g., [1], [8], or Wolfram MathWorld, Wikipedia).

The range of the operator. Let $h \in L^2(\mathbb{H})$ be such that $\mathbf{M}^{-1}[h] \in L^2(\mathbb{C}_+)$ is perpendicular to the range of the above operator. From the proof of Theorem 5.1, we see that this is the same as requiring that

$$\big\langle \mathbf{M}^{-1}[h], \mathbf{M}\boldsymbol{\Delta} F + \tfrac{\mathrm{i}}{2}(\boldsymbol{\partial} + \bar{\boldsymbol{\partial}})F \big\rangle_{L^2(\mathbb{C}_+)} = 0, \qquad F \in C_0^\infty(\mathbb{C}_+).$$

By dualizing we see that this is the same as

$$\big\langle \boldsymbol{\Delta} h + \tfrac{\mathrm{i}}{2}(\boldsymbol{\partial} + \bar{\boldsymbol{\partial}})\mathbf{M}^{-1}[h], F \big\rangle_{L^2(\mathbb{C}_+)} = 0, \qquad F \in C_0^\infty(\mathbb{C}_+),$$

that is,

$$\boldsymbol{\Delta} h + \frac{\mathrm{i}}{2}(\boldsymbol{\partial} + \bar{\boldsymbol{\partial}})\mathbf{M}^{-1}[h] = 0.$$

Since

$$\boldsymbol{\partial} + \bar{\boldsymbol{\partial}} = \frac{\partial}{\partial x},$$

this amounts to the differential equation

$$y\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)h + 2\mathrm{i}\frac{\partial}{\partial x}h = 0. \tag{6.1}$$

Lemma 6.1. *A function $h \in L^2(\mathbb{H})$ solves the partial differential equation* (6.1) *in $\mathbb{C}_+$ if and only if $\mathbf{M}^{-1}[h] \in \mathrm{conj}(A^2(\mathbb{C}_+))$.*

Proof. Let

$$\widehat{h}(\xi, y) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}x\xi} h(x, y)\,\mathrm{d}x$$

denote the partial Fourier transform with respect to the x variable. An application of the partial Fourier transform to the differential equation (6.1) yields

$$y\Big(-\xi^2 + \frac{\partial^2}{\partial y^2}\Big)\widehat{h}(\xi, y) - 2\xi\widehat{h}(\xi, y) = 0,$$

that is,

$$\frac{\partial^2}{\partial y^2}\widehat{h}(\xi, y) - \Big(\xi^2 + \frac{2\xi}{y}\Big)\widehat{h}(\xi, y) = 0. \tag{6.2}$$

Next, we put

$$H(\xi,t)=\widehat{h}\Big(\xi,\frac{t}{2|\xi|}\Big),$$

and see that (6.2) becomes

$$\frac{\partial^2}{\partial t^2}H(\xi,t)-\Big(\frac{1}{4}+\frac{\mathrm{sgn}(\xi)}{t}\Big)H(\xi,t)=0. \tag{6.3}$$

The requirement that $h\in L^2(\mathbb{H})$ amounts to

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}|H(\xi,t)|^2\,\frac{|\xi|\mathrm{d}\xi\mathrm{d}t}{t^2}<+\infty. \tag{6.4}$$

The differential equation (6.3) is of Whittaker type. It is well-known that the general solution to the ordinary differential equation

$$\frac{d^2}{\partial t^2}X(t)-\Big(\frac{1}{4}+\frac{1}{t}\Big)X(t)=0$$

is of the form

$$X(t)=A_1t\mathrm{e}^{t/2}+B_1\,t\mathrm{e}^{-t/2}\int_0^{+\infty}\mathrm{e}^{-t\theta}\frac{\theta}{1+\theta}\,\mathrm{d}\theta,$$

where A_1,B_1 are constants, while the general solution to the ordinary differential equation

$$\frac{d^2}{\partial t^2}Y(t)-\Big(\frac{1}{4}-\frac{1}{t}\Big)Y(t)=0$$

is of the form

$$Y(t)=A_2\mathrm{e}^{-t/2}\Big(1-t\log t-t\int_0^t\frac{\mathrm{e}^\theta-1-\theta}{\theta^2}\mathrm{d}\theta\Big)+B_2\,t\mathrm{e}^{-t/2},$$

where A_2,B_2 are constants. It follows that $H(\xi,t)$ must have form

$$H(\xi,t)=A_1(\xi)t\mathrm{e}^{t/2}+B_1(\xi)\,t\mathrm{e}^{-t/2}\int_0^{+\infty}\mathrm{e}^{-t\theta}\frac{\theta}{1+\theta}\,\mathrm{d}\theta,\qquad \xi>0,$$

and

$$H(\xi,t)=A_2(\xi)\mathrm{e}^{-t/2}\Big(1-t\log t-t\int_0^t\frac{\mathrm{e}^\theta-1-\theta}{\theta^2}\mathrm{d}\theta\Big)+B_2(\xi)\,t\mathrm{e}^{-t/2},\qquad \xi<0.$$

A careful analysis of the behavior of these solutions as $t\to0^+$ and $t\to+\infty$ shows that (6.4) is impossible unless $A_1(\xi)=B_1(\xi)=A_2(\xi)=0$, in which case

$$H(\xi,t)=0,\qquad \xi>0,$$

and

$$H(\xi,t)=B_2(\xi)\,t\mathrm{e}^{-t/2},\qquad \xi<0.$$

The function $B_2(\xi)$ must then satisfy

$$\int_{-\infty}^0|\xi|\,|B_2(\xi)|^2\mathrm{d}\xi<+\infty,$$

and the partial Fourier transform $\widehat{h}$ takes the form

$$\widehat{h}(\xi,y)=2|\xi|yB_2(\xi)\,\mathrm{e}^{y\xi}\,1_{]-\infty,0]}(\xi).$$

This form of $\widehat{h}$ is equivalent to the assertion that $\mathbf{M}^{-1}[h]\in\mathrm{conj}(A^2(\mathbb{C}_+))$. □

We now obtain the closure of the range of the operator.

Proposition 6.2. *The closure of the range of*

$$\mathbf{M}+\tfrac{\mathrm{i}}{2}\big[\mathbf{C}^{\downarrow}+\bar{\mathbf{C}}^{\downarrow}\big]:\,L^2(\mathbb{H}^*)\to L^2(\mathbb{C}_+)$$

equals $L^2(\mathbb{C}_+)\ominus\mathrm{conj}(A^2(\mathbb{C}_+))$.

Remark 6.3. One can show that the range of the operator is closed.

The null space of the operator. We turn to the study of the kernel of the operator. So, given $f \in L^2(\mathbb{H}^*)$, we want to know what the solutions to

$$\mathbf{M}[f] + \frac{\mathrm{i}}{2}\big[\mathbf{C}^{\downarrow}[f] + \bar{\mathbf{C}}^{\downarrow}[f]\big] = 0 \tag{6.5}$$

look like. Let $F \in L^2(\mathbb{H})$ be the associated function

$$F = \bar{\mathbf{C}}^{\uparrow}\mathbf{C}^{\downarrow}[f] = \mathbf{C}^{\uparrow}\bar{\mathbf{C}}^{\downarrow}[f].$$

Then $\boldsymbol{\Delta} F = f$, and (6.5) takes the form

$$\mathbf{M}[\boldsymbol{\Delta} F] + \frac{\mathrm{i}}{2}[\boldsymbol{\partial} F + \bar{\boldsymbol{\partial}} F] = 0. \tag{6.6}$$

By Lemma 6.1, we find that this happens if and only if $\mathbf{M}^{-1}[F] \in \operatorname{conj}(A^2(\mathbb{C}_+))$. From this, we quickly derive the following characterization of the null space.

Proposition 6.4. *The null space of the operator*

$$\mathbf{M} + \frac{\mathrm{i}}{2}\big[\mathbf{C}^{\downarrow} + \bar{\mathbf{C}}^{\downarrow}\big] : L^2(\mathbb{H}^*) \to L^2(\mathbb{C}_+)$$

equals $\operatorname{conj}(A^2(\mathbb{H}^*))$*, the space of conjugate-holomorphic functions in* $L^2(\mathbb{H}^*)$*.*

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